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## Representing congruence lattices of lattices with partial unary operations as congruence lattices of lattices. II. Interval ordering

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### Abstract

In Part I of this paper, we introduced a method of making two isomorphic intervals of a bounded lattice congruence equivalent. In this paper, we make one interval dominate another one.

Let  $L$  be a bounded lattice, let  $[a, b]$  and  $[c, d]$  be intervals of  $L$ , and let  $\varphi$  be a homomorphism of  $[a, b]$  onto  $[c, d]$ . We construct a bounded (convex) extension  $K$  of  $L$  such that a congruence  $\Theta$  of  $L$  has an extension to  $K$  iff  $x \equiv y(\Theta)$  implies that  $x\varphi \equiv y\varphi(\Theta)$ , for  $a \leq x \leq y \leq b$ , in which case,  $\Theta$  has a unique extension to  $K$ .

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This result presents a lattice  $K$  whose congruence lattice is derived from the congruence lattice of  $L$  in a new way, different from the one presented in Part I.

The main technical innovation is the *2/3-Boolean triple construction*, which owes its origin to the Boolean triple construction of G. Grätzer and F. Wehrung.

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## 1. Introduction

To keep this paper short, we assume that the reader is familiar with [6], Part I of this paper. Recall that the lattice  $K$  is an *extension* of the lattice  $L$ , if  $L$  is a sublattice of  $K$ . The lattice  $K$  is a *convex extension* of the lattice  $L$ , if  $L$  is a convex sublattice of  $K$ . A *convex embedding* is defined analogously.

In Part I, we constructed a “magic wand”—as a (convex) extension—that will force that  $a \equiv b$  be equivalent to  $c \equiv d$  in a bounded lattice  $L$ . In this paper, we construct a “one-directional magic wand”:  $a \equiv b$  implies that  $c \equiv d$ .

### 1.1. One surjective homomorphism

Let  $L$  be a bounded lattice, let  $[a, b]$  and  $[c, d]$  be intervals of  $L$ , let  $\varphi$  be a homomorphism of  $[a, b]$  onto  $[c, d]$ . We can consider  $\varphi$  as a partial unary operation. Let us call a congruence  $\Theta$  of  $L$  a  $\vec{\varphi}$ -congruence iff  $\Theta$  satisfies the Substitution Property with respect to the partial unary operation  $\varphi$ , that is,  $x \equiv y(\Theta)$  implies that  $x\varphi \equiv y\varphi(\Theta)$ , for all  $x, y \in [a, b]$ . (The symbol  $\rightarrow$  on top of  $\varphi$  signifies that the partial operation goes only one way.) Let  $L_{\vec{\varphi}}$  denote the partial algebra obtained from  $L$  by adding the partial operation  $\varphi$ . Thus, a congruence relation of  $L_{\vec{\varphi}}$  is the same as a  $\vec{\varphi}$ -congruence of  $L$ . We call an extension  $K$  of  $L$  a  $\vec{\varphi}$ -congruence-preserving extension of  $L$ , if a congruence of  $L$  extends to  $K$  iff it is a  $\vec{\varphi}$ -congruence and every  $\vec{\varphi}$ -congruence of  $L$  has *exactly one* extension to  $K$ . If  $K$  is a  $\vec{\varphi}$ -congruence-preserving extension of  $L$ , then the congruence lattice of the partial algebra  $L_{\vec{\varphi}}$  is isomorphic to the congruence lattice of the lattice  $K$ .

Let us call  $\varphi$  *algebraic* iff there is a unary algebraic function  $\mathbf{p}(x)$  (that is,  $\mathbf{p}(x)$  is obtained from a lattice polynomial by substituting all but one of the variables by elements of  $L$ ) such that  $x\varphi = \mathbf{p}(x)$ , for all  $x \in [a, b]$ .

We prove the following result:

**Theorem 1.** *Let  $L$  be a bounded lattice, let  $[a, b]$  and  $[c, d]$  be intervals of  $L$ , and let  $\varphi$  be a homomorphism of  $[a, b]$  onto  $[c, d]$ . Then  $L$  has a  $\vec{\varphi}$ -congruence-preserving convex extension into a bounded lattice  $K$  such that  $\varphi$  is algebraic in  $K$ . In particular, the congruence lattice of the partial algebra  $L_{\vec{\varphi}}$  is isomorphic to the congruence lattice of the bounded lattice  $K$ . If  $L$  is finite, then  $K$  can be constructed as a finite lattice.*

So  $K$  realizes the “magic wand”:  $a \equiv b$  implies that  $c \equiv d$ .

Note that Theorem 1 of this paper implies Theorem 1 of Part I, which states that if  $L$  is a bounded lattice,  $[a, b]$  and  $[c, d]$  are intervals of  $L$ , and  $\varphi$  is an isomorphism of  $[a, b]$  and  $[c, d]$ , then  $L$  has a bounded extension  $K$  such that a congruence  $\Theta$  of  $L$  extends to  $K$  iff  $x \equiv y(\Theta)$  is equivalent to  $x\varphi \equiv y\varphi(\Theta)$ , for all  $x, y \in [a, b]$ ; in which case,  $\Theta$  has exactly one extension to  $K$ . We obtain such an extension  $K$  by applying Theorem 1 of this paper to  $L$  to obtain a convex extension  $K_1$  for  $\varphi$ , and then applying it to  $K_1$ , the intervals  $[a, b]$  and  $[c, d]$  (which remain intervals in  $K_1$  because of the convexity of the extension) and  $\varphi^{-1}$  to obtain  $K$ . Observe that in this application,  $\varphi$  is an isomorphism, while in this paper,  $\varphi$  is only an onto homomorphism.

### 1.2. Many surjective homomorphisms

Let  $L$  be a bounded lattice, and for  $i \in I$ , let  $\varphi_i$  be a homomorphism of the interval  $[a_i, b_i]$  onto the interval  $[c_i, d_i]$ . Let

$$\Phi = \{\varphi_i \mid i \in I\},$$

and let  $L_\Phi$  denote the partial algebra obtained from  $L$  by adding the partial operations  $\varphi_i$ , for  $i \in I$ . Let us call a congruence  $\Theta$  of  $L$  a  $\Phi$ -congruence iff  $\Theta$  satisfies the Substitution Property with respect to the partial unary operations  $\varphi_i$ ,  $i \in I$ , that is,  $x \equiv y(\Theta)$  implies that  $x\varphi_i \equiv y\varphi_i(\Theta)$ , for all  $x, y \in [a_i, b_i]$  and  $i \in I$ . Thus, a congruence relation of  $L_\Phi$  is the same as a  $\Phi$ -congruence of  $L$ . We call  $K$  a  $\Phi$ -congruence-preserving extension of  $L$ , if a congruence of  $L$  extends to  $K$  iff it is a  $\Phi$ -congruence of  $L$  and every  $\Phi$ -congruence of  $L$  has *exactly one* extension to  $K$ .

**Theorem 2.** *Let  $L$  be a bounded lattice, let  $\Phi$  be given as above. Then the partial algebra  $L_\Phi$  has a  $\Phi$ -congruence-preserving convex extension into a lattice  $K$  such that all  $\varphi_i$ ,  $i \in I$ , are algebraic in  $K$ . In particular, the congruence lattice of the partial algebra  $L_\Phi$  is isomorphic to the congruence lattice of the lattice  $K$ .*

Theorem 2 of this paper easily implies Theorems 2 and 3 of Part I, with one important difference: In Theorem 2 of Part I, we obtain a *bounded* lattice  $K$ . Unfortunately, we do not know how to ensure the  $K$  be bounded in Theorem 2 of this paper; recall that we did not know how to obtain a bounded  $K$  in Theorem 3 of Part I. This shows why the two parts of this paper would be difficult to merge: To obtain Theorem 2 of Part I, we would have to present the construction of Part I; the construction in this paper would be of no help.

### 1.3. Outline

The proofs are basically the same as in Part I, except that the lattice  $B$  of Part I is radically different from the lattice  $B$  of this paper. This new  $B$  is constructed using the “2/3-Boolean triple construction” described in Section 2. We prove Theorem 1 in Section 3; we just provide the arguments that are necessary to change the proof of Part I.

Theorem 2 is proved in Section 4. Finally, in Section 5, we discuss some applications and a few open problem.

We use the standard notation, as in [2] and [6].

## 2. The 2/3-Boolean triple construction

### 2.1. The Boolean triple construction

G. Grätzer and F. Wehrung [7], for a lattice  $P$ , introduced Boolean triples: the element  $\langle x, y, z \rangle \in P^3$  is called a *Boolean triple* iff

$$x = (x \vee y) \wedge (x \vee z),$$

$$y = (y \vee x) \wedge (y \vee z),$$

$$z = (z \vee x) \wedge (z \vee y).$$

They proved that  $M_3\langle P \rangle$ , the set of all Boolean triples partially ordered componentwise, is a lattice, in fact, a congruence-preserving extension of  $P$ .

### 2.2. The $N_6\langle P \rangle$ construction

Let  $N_6 = \{o, p, q_1, q_2, r, i\}$  denote the six-element lattice depicted in Fig. 1, with  $o$  the zero,  $i$  the unit element,  $p, q_1, q_2$  the atoms, satisfying the relations  $q_1 \vee q_2 = r$ ,  $p \wedge q_1 = p \wedge q_2 = p \wedge r = o$ , and  $p \vee q_1 = p \vee q_2 = p \vee r = i$ .

In this paper, for a bounded lattice  $P$ , we introduce 2/3-Boolean triples: the element  $\langle x, y, z \rangle \in P^3$  is called a *2/3-Boolean triple* iff

$$y = (y \vee x) \wedge (y \vee z),$$

$$z = (z \vee x) \wedge (z \vee y).$$

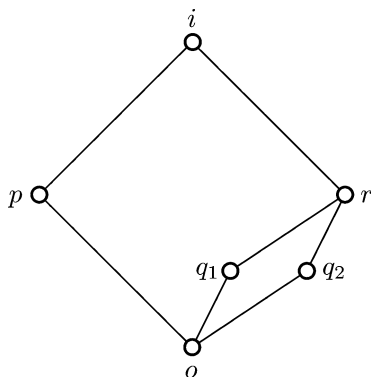


Fig. 1. The lattice  $N_6$ .

We retain two out of three conditions that define Boolean triples, hence the name. The notation  $N_6\langle P \rangle$  follows the notational convention of G. Grätzer and M. Greenberg [3] and G. Grätzer and F. Wehrung [8]; the role of  $N_6$  in this construction is clarified by Corollary 2 and Lemma 3.

We denote by  $N_6\langle P \rangle$  the set of all 2/3-Boolean triples partially ordered componentwise. In this section, we prove that  $N_6\langle P \rangle$  is a lattice and describe the congruences of this lattice.

**Lemma 1.**  $N_6\langle P \rangle$  is a closure system in  $P^3$ ; let  $\overline{\langle x, y, z \rangle}$  denote the closure of  $\langle x, y, z \rangle \in P^3$  and call it the 2/3-Boolean closure of  $\langle x, y, z \rangle \in P^3$ . Then

$$\overline{\langle x, y, z \rangle} = \langle x, (y \vee x) \wedge (y \vee z), (z \vee x) \wedge (z \vee y) \rangle.$$

**Proof.** In this proof, let  $\bar{y} = (y \vee x) \wedge (y \vee z)$  and  $\bar{z} = (z \vee x) \wedge (z \vee y)$ . We have to verify that  $\overline{\langle x, y, z \rangle} = \langle x, \bar{y}, \bar{z} \rangle$  is the closure of  $\langle x, y, z \rangle$ .

The triple  $\langle x, \bar{y}, \bar{z} \rangle$  is 2/3-Boolean closed. Indeed,  $y \leq \bar{y}$ , so  $x \vee y = x \vee \bar{y}$ . Also,  $z \leq \bar{z}$ , so  $\bar{y} \vee \bar{z} = y \vee z$ . Therefore,

$$(\bar{y} \vee x) \wedge (\bar{y} \vee \bar{z}) = \bar{y},$$

verifying the first half of the definition of 2/3-Boolean triples; the second half is proved similarly.

So  $\langle x, y, z \rangle \leq \langle x, \bar{y}, \bar{z} \rangle \in N_6\langle P \rangle$ . To prove that  $N_6\langle P \rangle$  is a closure system in  $P^3$  and that  $\langle x, \bar{y}, \bar{z} \rangle$  is the closure of  $\langle x, y, z \rangle$ , it suffices to verify that if  $\langle x_1, y_1, z_1 \rangle \in N_6\langle P \rangle$  and  $\langle x, y, z \rangle \leq \langle x_1, y_1, z_1 \rangle$ , then  $\langle x, \bar{y}, \bar{z} \rangle \leq \langle x_1, y_1, z_1 \rangle$ , which is obvious.  $\square$

**Corollary 2.**  $N_6\langle P \rangle$  is a lattice. Meet is componentwise and join is the closure of the componentwise join. Moreover,  $N_6\langle P \rangle$  has a spanning  $N_6$  (see Fig. 2):

$$\{o = \langle 0, 0, 0 \rangle, p = \langle 1, 0, 0 \rangle, q_1 = \langle 0, 1, 0 \rangle, q_2 = \langle 0, 0, 1 \rangle, r = \langle 0, 1, 1 \rangle, i = \langle 1, 1, 1 \rangle\}.$$

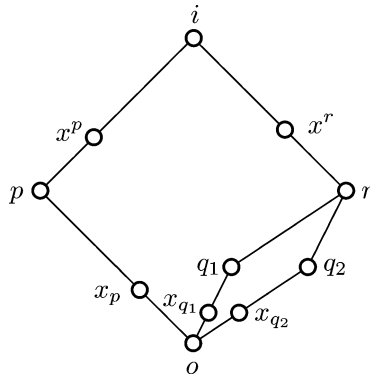


Fig. 2. Illustrating  $N_6\langle P \rangle$ .

**Lemma 3.**

(i) The interval  $[o, p]$  of  $N_6\langle P \rangle$  is isomorphic to  $P$  under the isomorphism

$$x_p = \langle x, 0, 0 \rangle \mapsto x, \quad x \in P.$$

(ii) The interval  $[o, q_1]$  of  $N_6\langle P \rangle$  is isomorphic to  $P$  under the isomorphism

$$x_{q_1} = \langle 0, x, 0 \rangle \mapsto x, \quad x \in P.$$

(iii) The interval  $[o, q_2]$  of  $N_6\langle P \rangle$  is isomorphic to  $P$  under the isomorphism

$$x_{q_2} = \langle 0, 0, x \rangle \mapsto x, \quad x \in P.$$

(iv) The interval  $[p, i]$  of  $N_6\langle P \rangle$  is isomorphic to  $P$  under the isomorphism

$$x^p = \langle 1, x, x \rangle \mapsto x, \quad x \in P.$$

(v) The interval  $[o, r]$  of  $N_6\langle P \rangle$  is isomorphic to  $P^2$  under the isomorphism

$$\langle 0, x, y \rangle \mapsto \langle x, y \rangle, \quad x, y \in P.$$

(vi) The interval  $[r, i]$  of  $N_6\langle P \rangle$  is isomorphic to  $P$  under the isomorphism

$$x^r = \langle x, 1, 1 \rangle \mapsto x, \quad x \in P.$$

**Proof.** By trivial computation. For instance, to prove (iv), observe that  $\langle 1, x, y \rangle$  is closed iff  $x = y$ .  $\square$

For the five isomorphic copies of  $P$  in  $N_6\langle P \rangle$ , we use the notation:

$$P_p = [o, p],$$

$$P_{q_1} = [o, q_1],$$

$$P_{q_2} = [o, q_2],$$

with zero  $o$  and unit elements,  $p, q_1, q_2$ , respectively, and

$$P^p = [p, i],$$

$$P^r = [r, i].$$

with unit 1 and zero elements,  $p, r$ , respectively.

### 2.3. The congruences of $N_6\langle P \rangle$

We describe the congruence structure of  $N_6\langle P \rangle$  based on the following decomposition of elements.

**Lemma 4.** Every  $\alpha \in N_6\langle P \rangle$  has a decomposition

$$\alpha = (\alpha \wedge p) \vee (\alpha \wedge q_1) \vee (\alpha \wedge q_2),$$

where  $\alpha \wedge p \in P_p$ ,  $\alpha \wedge q_1 \in P_{q_1}$ , and  $\alpha \wedge q_2 \in P_{q_2}$ .

**Proof.** Indeed, the componentwise join of the right side equals  $\alpha$ .  $\square$

For a congruence  $\Psi$  of  $N_6\langle P \rangle$ , let  $\Psi_p$  denote the restriction of  $\Psi$  to  $P_p$ , same for  $\Psi_{q_1}$  and  $\Psi_{q_2}$ . Let  $\hat{\Psi}_p$  denote  $\Psi_p$  regarded as a congruence of  $P$ ; same for  $\hat{\Psi}_{q_1}$  and  $\hat{\Psi}_{q_2}$ . Similarly, let  $\Psi^p$  and  $\Psi^r$  denote the restriction of  $\Psi$  to  $P^p$  and  $P^r$ , respectively, and let  $\hat{\Psi}^p$  and  $\hat{\Psi}^r$  denote the corresponding congruences of  $P$ . Then we obtain:

**Lemma 5.** Let  $\alpha, \alpha' \in N_6\langle P \rangle$  and  $\Psi \in \text{Con } N_6\langle P \rangle$ . Then

$$\alpha \equiv \alpha' \ (\Psi)$$

iff

$$\alpha \wedge p \equiv \alpha' \wedge p \ (\Psi_p),$$

$$\alpha \wedge q_1 \equiv \alpha' \wedge q_1 \ (\Psi_{q_1}),$$

$$\alpha \wedge q_2 \equiv \alpha' \wedge q_2 \ (\Psi_{q_2}).$$

**Proof.** This is clear from Lemma 4.  $\square$

**Lemma 6.** Let  $\alpha = \langle x, y, z \rangle$ ,  $\alpha' = \langle x', y', z' \rangle \in N_6\langle P \rangle$  and  $\Psi \in \text{Con } N_6\langle P \rangle$ . Then

$$\alpha \equiv \alpha' \ (\Psi)$$

iff

$$x \equiv x' \ (\hat{\Psi}_p),$$

$$y \equiv y' \ (\hat{\Psi}_{q_1}),$$

$$z \equiv z' \ (\hat{\Psi}_{q_2}).$$

**Proof.** This is clear from Lemma 5.  $\square$

Now we have the tools to describe the congruences. The next four lemmas provide the description.

**Lemma 7.** Let  $\Psi \in \text{Con } N_6\langle P \rangle$ . Then  $\hat{\Psi}_{q_1} = \hat{\Psi}_{q_2}$  in  $P$ .

**Proof.** Indeed, if  $u \equiv u'(\hat{\Psi}_{q_1})$ , then  $\langle 0, u, 0 \rangle \equiv \langle 0, u', 0 \rangle(\Psi_{q_1})$ , so  $\langle 0, u, 0 \rangle \equiv \langle 0, u', 0 \rangle(\Psi)$ . Therefore,

$$\begin{aligned} \langle 0, 0, u \rangle &= (\langle 0, u, 0 \rangle \vee \langle 1, 0, 0 \rangle) \wedge \langle 0, 0, 1 \rangle \\ &\equiv (\langle 0, u', 0 \rangle \vee \langle 1, 0, 0 \rangle) \wedge \langle 0, 0, 1 \rangle = \langle 0, 0, u' \rangle(\Psi). \end{aligned}$$

We conclude that  $\langle 0, 0, u \rangle \equiv \langle 0, 0, u' \rangle(\Psi_{q_2})$ , that is,  $u \equiv u'(\hat{\Psi}_{q_2})$ , proving that  $\hat{\Psi}_{q_1} \leq \hat{\Psi}_{q_2}$ . By symmetry,  $\hat{\Psi}_{q_1} \geq \hat{\Psi}_{q_2}$ , so  $\hat{\Psi}_{q_1} = \hat{\Psi}_{q_2}$ .  $\square$

**Lemma 8.** For  $\Psi \in \text{Con } N_6\langle P \rangle$ , the congruence inequality  $\hat{\Psi}_p \leq \hat{\Psi}_{q_2}$  holds.

**Proof.** Indeed, if  $u \equiv u'(\hat{\Psi}_p)$ , then  $\langle u, 0, 0 \rangle \equiv \langle u', 0, 0 \rangle(\Psi_p)$ . Therefore,

$$\begin{aligned} \langle 0, 0, u \rangle &= \langle u, 1, u \rangle \wedge \langle 0, 0, 1 \rangle = (\langle u, 0, 0 \rangle \vee \langle 0, 1, 0 \rangle) \wedge \langle 0, 0, 1 \rangle \\ &\equiv (\langle u', 0, 0 \rangle \vee \langle 0, 1, 0 \rangle) \wedge \langle 0, 0, 1 \rangle \\ &= \langle u', 1, u' \rangle \wedge \langle 0, 0, 1 \rangle = \langle 0, 0, u' \rangle(\Psi), \end{aligned}$$

that is,  $u \equiv u'(\hat{\Psi}_{q_2})$ , proving that  $\hat{\Psi}_p \leq \hat{\Psi}_{q_2}$ .  $\square$

**Lemma 9.** Let  $\Theta \leq \Phi \in \text{Con } P$ . Then there is a unique  $\Psi \in \text{Con } N_6\langle P \rangle$ , such that  $\hat{\Psi}_p = \Theta$  and  $\hat{\Psi}_{q_1} = \hat{\Psi}_{q_2} = \Phi$ .

**Proof.** The uniqueness follows from the previous lemmas. To prove the existence, for  $\Theta \leq \Phi \in \text{Con } P$ , define a congruence  $\Psi$  on  $N_6\langle P \rangle$  by

$$\langle x, y, z \rangle \equiv \langle x', y', z' \rangle(\Psi)$$

iff

$$x \equiv x'(\Theta),$$

$$y \equiv y'(\Phi),$$

$$z \equiv z'(\Phi).$$

It is obvious that  $\Psi$  is an equivalence relation and it satisfies the Substitution Property for meet. To verify the Substitution Property for join, let  $\langle x, y, z \rangle \equiv \langle x', y', z' \rangle(\Psi)$  and let  $\langle u, v, w \rangle \in N_6\langle P \rangle$ . Then

$$\begin{aligned} \langle x, y, z \rangle \vee \langle u, v, w \rangle &= \overline{\langle x \vee u, y \vee v, z \vee w \rangle} \\ &= \langle x \vee u, (x \vee y \vee u \vee v) \wedge (y \vee z \vee v \vee w), (x \vee z \vee u \vee w) \wedge (y \vee z \vee v \vee w) \rangle. \end{aligned}$$



Similarly,

$$\begin{aligned} & \langle x', y', z' \rangle \vee \langle u, v, w \rangle \\ &= \langle x' \vee u, (x' \vee y' \vee u \vee v) \wedge (y' \vee z' \vee v \vee w), (x' \vee z' \vee u \vee w) \wedge (y' \vee z' \vee v \vee w) \rangle. \end{aligned}$$

Since  $x \equiv x'(\Theta)$ , we also have

$$x \vee u \equiv x' \vee u (\Theta).$$

From  $x \equiv x'(\Theta)$  and  $\Theta \leq \Phi$ , it follows that  $x \equiv x'(\Phi)$ . Also,  $y \equiv y'(\Phi)$ , so  $x \vee y \equiv x' \vee y'(\Phi)$ . Therefore,  $x \vee y \vee u \vee v \equiv x' \vee y' \vee u \vee v(\Phi)$ . Similarly (or even simpler),  $y \vee z \vee v \vee w \equiv y' \vee z' \vee v \vee w(\Phi)$ . Meeting the last two congruences, we obtain that

$$(x \vee y \vee u \vee v) \wedge (y \vee z \vee v \vee w) \equiv (x' \vee y' \vee u \vee v) \wedge (y' \vee z' \vee v \vee w) (\Phi).$$

Similarly,

$$(x \vee z \vee u \vee w) \wedge (y \vee z \vee v \vee w) \equiv (x' \vee z' \vee u \vee w) \wedge (y' \vee z' \vee v \vee w) (\Phi).$$

The last three displayed equations verify that

$$\langle x, y, z \rangle \vee \langle u, v, w \rangle \equiv \langle x', y', z' \rangle \vee \langle u, v, w \rangle (\Phi). \quad \square$$

Now note that for  $x, y \in P$  and congruence  $\Psi$  of  $P$ ,

$$x_p \equiv y_p(\Psi) \quad \text{iff} \quad x^r \equiv y^r(\Psi)$$

and

$$x^p \equiv y^p(\Psi) \quad \text{iff} \quad x_{q_1} \equiv y_{q_1}(\Psi).$$

It follows that  $\hat{\Psi}^p = \hat{\Psi}_{q_1} = \hat{\Psi}_{q_2}$  and  $\hat{\Psi}^r = \hat{\Psi}_p$ , so Lemmas 7–9 can be restated as follows.

**Corollary 10.** *There is a one-to-one correspondence between the congruences of  $N_6\langle P \rangle$  and pairs of congruences  $\Theta \leq \Phi$  of  $\text{Con } P$ , defined by*

$$\Psi \mapsto \langle \hat{\Psi}^r, \hat{\Psi}^p \rangle.$$

Lemma 9 and Corollary 10 can be proved using lattice tensor products introduced in G. Grätzer and F. Wehrung [8]. The lattice  $N_6\langle P \rangle$  is isomorphic to the lattice tensor product of  $N_6$  and  $P$ ; an isomorphism is given by

$$\langle x, y, z \rangle \mapsto (p \boxtimes x) \vee (q_1 \boxtimes x) \vee (q_2 \boxtimes x).$$

The results in [8] can be used to verify these lemmas.

In the special case when one of the factors is finite, G. Grätzer and M. Greenberg [3] provide a much more elementary approach to lattice tensor products by providing a coordinatization. Our definition of  $N_6\langle P \rangle$  is based on this coordinatization, especially, on Section 3 of [3].

By the main result of G. Grätzer and M. Greenberg [4] (using the notation of [4]),

$$\begin{aligned}\text{Con } N_6\langle P \rangle &\cong (\text{Con } N_6)\langle \text{Con } P \rangle \\ &\cong C_3\langle \text{Con } P \rangle \\ &\cong \{ \langle \Theta, \Phi \rangle \mid \Theta \leq \Phi \text{ in } \text{Con } P \},\end{aligned}$$

where  $C_3$  is the three-element chain. But there is more in [4]; the isomorphism is explicitly exhibited. Let  $\mathcal{F}: \text{Con } N_6\langle P \rangle \rightarrow (\text{Con } N_6)\langle \text{Con } P \rangle$  be the isomorphism. Then by definition (see Eq. (11) in [4]),  $\mathcal{F}(\Psi)(\Theta_{N_6}(u, v)) = \Psi_{uv}$ . Therefore,  $\mathcal{F}: \Psi \mapsto \langle \Psi_{0p}, \Psi_{0q_1} \rangle$ . In other words, each congruence of  $N_6\langle P \rangle$  is obtained by taking two congruences  $\Theta \leq \Phi$  of  $P$ , imposing  $\Theta$  on the interval  $[0, p] \subseteq N_6\langle P \rangle$ , and  $\Phi$  on the interval  $[0, q_1] \subseteq N_6\langle P \rangle$ . This verifies again Lemma 9 and Corollary 10.

#### 2.4. An algebraic function on $N_6\langle P \rangle$

The inequality  $\hat{\Psi}^r \leq \hat{\Psi}^p$  can be established in a stronger form by exhibiting an algebraic function  $\mathbf{r}(x)$  on  $N_6\langle P \rangle$  such that  $\mathbf{r}(u^r) = u^p$ , for  $u \in P$ . We now proceed to exhibit  $\mathbf{r}(x)$ , which we shall need in Section 3.

**Lemma 11.** *There is an algebraic function  $\mathbf{r}(x)$  on  $N_6\langle P \rangle$  such that  $\mathbf{r}(u^r) = u^p$ , for  $u \in P$ .*

**Proof.** Define

$$\mathbf{r}(x) = (((x \wedge p) \vee q_1) \wedge q_2) \vee p.$$

Indeed, if  $u \in P$ , then  $u^r = \langle u, 1, 1 \rangle$  and so

$$\begin{aligned}\mathbf{r}(u^r) &= \mathbf{r}(\langle u, 1, 1 \rangle) \\ &= (((\langle u, 1, 1 \rangle \wedge \langle 1, 0, 0 \rangle) \vee \langle 0, 1, 0 \rangle) \wedge \langle 0, 0, 1 \rangle) \vee \langle 1, 0, 0 \rangle \\ &= ((\langle u, 0, 0 \rangle \vee \langle 0, 1, 0 \rangle) \wedge \langle 0, 0, 1 \rangle) \vee \langle 1, 0, 0 \rangle \\ &= (\overline{\langle u, 1, 0 \rangle} \wedge \langle 0, 0, 1 \rangle) \vee \langle 1, 0, 0 \rangle \\ &= (\langle u, 1, u \rangle \wedge \langle 0, 0, 1 \rangle) \vee \langle 1, 0, 0 \rangle \\ &= \langle 0, 0, u \rangle \vee \langle 1, 0, 0 \rangle \\ &= \overline{\langle 1, 0, u \rangle} \\ &= \langle 1, u, u \rangle = u^p,\end{aligned}$$

as required.  $\square$

### 2.5. A quotient of $N_6\langle P \rangle$

Actually, to prove Theorem 1, we need not the lattice  $N_6\langle P \rangle$ , but a quotient thereof, which we now proceed to construct.

Let  $P$  and  $Q$  be bounded lattices of more than one element, and let

$$\varphi: P \rightarrow Q$$

be a homomorphism of  $P$  onto  $Q$ . We denote by  $\ker \varphi$  the *kernel* of this homomorphism,  $\ker \varphi \in \text{Con } P$ .

By Corollary 10, there is a unique congruence  $\Psi$  of  $N_6\langle P \rangle$  corresponding to the congruence pair  $\omega \leq \ker \varphi$  of  $P$ . Define

$$B = N_6\langle P \rangle / \Psi.$$

It is useful to note that  $B$  can be represented as

$$\{ \langle x, y, z \rangle \in P \times Q \times Q \mid y = (y \vee x\varphi) \wedge (y \vee z) \text{ and } z = (z \vee x\varphi) \wedge (z \vee y) \}.$$

By the Second Isomorphism Theorem (see, e.g., [1]), there is a one-to-one correspondence between the congruences of  $B$  and congruence pairs  $\Theta \leq \Phi$  of  $P$  satisfying  $\ker \varphi \leq \Phi$ . For  $x \in N_6\langle P \rangle$ , let  $\bar{x}$  denote the congruence class  $[x]\Psi$ .

Using this notation, utilizing the results of this section, we state some important properties of the lattice  $B$ :

**Lemma 12.** *Let  $P$  and  $Q$  be bounded lattices and let  $\varphi: P \rightarrow Q$  be a homomorphism of  $P$  onto  $Q$ . Then there is a lattice  $B$ , with the following properties:*

- (i)  $B$  has a spanning sublattice  $\bar{o}, \bar{p}, \bar{q}_1, \bar{q}_2, \bar{r}, \bar{i}$  isomorphic to  $N_6$ .
- (ii) The interval  $[\bar{r}, \bar{i}]$  is isomorphic to  $P$  under the map  $x \mapsto \bar{x}^{\bar{r}}, x \in P$ .
- (iii) The interval  $[\bar{p}, \bar{i}]$  is isomorphic to  $Q$  under the map  $y \mapsto \bar{y}^{\bar{p}}, y \in Q$ , where  $x \in P$  with  $x\varphi = y$ .
- (iv) The congruences  $\Sigma$  of  $B$  are in one-to-one correspondence with pairs of congruences  $\langle \Theta, \Phi \rangle$ , where  $\Theta$  is a congruence of  $P$  and  $\Phi$  is a congruence of  $Q$  satisfying  $\Theta \leq \Phi\varphi^{-1}$ , where up to isomorphism,  $\Sigma$  restricted to  $[\bar{r}, \bar{i}]$  is  $\Theta$  and  $\Sigma$  restricted to  $[\bar{p}, \bar{i}]$  is  $\Phi$ .
- (v) There is an algebraic function  $\mathbf{r}(x)$  such that  $\mathbf{r}(\bar{u}^{\bar{r}}) = \bar{x}^{\bar{p}}$ , for  $u \in P$ , where  $x \in P$  with  $x\varphi = u$ .

### 3. Proof of Theorem 1

We proceed with the proof of Theorem 1 as in Part I. We can assume, without loss of generality, that  $[a, b]$  and  $[c, d]$  are nontrivial intervals, that is,  $a < b$  and  $c < d$ .

### 3.1. Constructing $K$

We take the four building blocks:  $A = M_3\langle L \rangle$ ,  $L_{a,b}$ ,  $B$ , and  $L_{c,d}$ , except that now  $B$  is not  $M_3\langle [a, b] \rangle$  but the lattice  $B = N_6\langle [a, b] \rangle / \Psi$  as described in Section 2.5 (especially in Lemma 12), constructed from  $P = [a, b]$ ,  $Q = [c, d]$ , and the homomorphism  $\varphi$  from  $[a, b]$  onto  $[c, d]$  given in the assumptions of Theorem 1.

We do three gluings.

*First gluing.* In  $B$ , we use the dual ideal

$$[\bar{r}] = \{ \overline{\langle x, 1, 1 \rangle} \mid a \leq x \leq b \}$$

(which is isomorphic to  $[a, b]$  since  $\Psi$  on  $[r, 1]$  is  $\omega$ ), while in  $L_{a,b}$  we utilize the ideal

$$I_{a,b} = \{ \langle 0, x, 0 \rangle \mid a \leq x \leq b \}$$

(which is obviously isomorphic to  $[a, b]$ ), and we consider the natural isomorphism

$$\varphi_1 : \overline{\langle x, 1, 1 \rangle}_B \mapsto \langle 0, x, 0 \rangle_{L_{a,b}}, \quad x \in [a, b],$$

between the dual ideal  $[\bar{r}]$  of  $B$  and the ideal  $I_{a,b}$  of  $L_{a,b}$  to glue  $B$  and  $L_{a,b}$  together to obtain the lattice  $U$ . (As in Part I, we use the following notation: to indicate whether a triple belongs to  $A = M_3\langle L \rangle$ ,  $L_{a,b}$ ,  $B$ , or  $L_{c,d}$ , we subscript the triple with  $A$ ,  $L_{a,b}$ ,  $B$ , or  $L_{c,d}$ , respectively.)

*Second gluing.* We glue  $L_{c,d}$  and  $A$  over the dual ideal

$$D_{c,d} = \{ \langle x, c, x \wedge c \rangle \mid x \in L \}$$

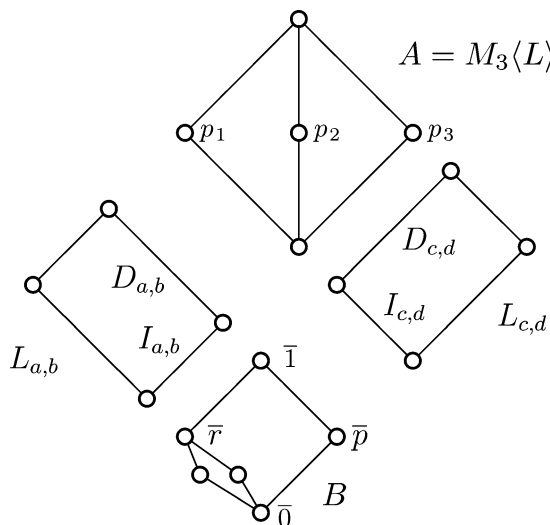


Fig. 3. The four building blocks of  $K$ .

of  $L_{c,d}$  and the ideal

$$(p_3] = \{ \langle 0, 0, x \rangle \mid x \in L \}$$

of  $A$ , with respect to the natural isomorphism

$$\varphi_2 : \langle x, c, x \wedge c \rangle_{L_{c,d}} \mapsto \langle 0, 0, x \rangle_A, \quad x \in L,$$

to obtain the lattice  $V$ .

*Final gluing.* In  $U$ , we define the dual ideal

$$D = [\bar{p}, \bar{1}] \cup D_{a,b},$$

which is the union of  $[\bar{p}, \bar{1}]$  and  $D_{a,b}$ , with the unit of  $[\bar{p}, \bar{1}]$  identified with the zero of  $D_{a,b}$ .

In  $V$ , we define the ideal

$$I = I_{c,d} \cup [0_A, p_1],$$

which is the union of  $I_{c,d}$  and  $[0_A, p_1]$ , with the unit of  $I_{c,d}$  identified with the zero of  $[0_A, p_1]$ .

Next we set up an isomorphism  $\psi : D \rightarrow I$ . Since

$$[\bar{p}, \bar{1}] = \{ \overline{\langle 1, x, x \rangle_B} \mid a \leq x \leq b \}$$

and

$$I_{c,d} = \{ \langle 0, x, 0 \rangle_{L_{c,d}} \mid c \leq x \leq d \},$$

we define  $\psi$  on  $[\bar{p}, \bar{1}]$  by

$$\psi : \overline{\langle 1, x, x \rangle_B} \mapsto \langle 0, x\varphi, 0 \rangle_{L_{c,d}},$$

where  $\varphi : [a, b] \rightarrow [c, d]$  is the isomorphism given in Theorem 1. We define  $\psi$  on  $D_{a,b}$  by

$$\psi : \langle x, b, x \wedge b \rangle_{L_{a,b}} \mapsto \langle x, 0, 0 \rangle_A, \quad x \in L.$$

It is clear that  $\psi : D \rightarrow I$  is well-defined and it is an isomorphism.

Finally, we construct the lattice  $K$  of Theorem 1 by gluing  $U$  over  $I$  with  $V$  over  $D$  with respect to the isomorphism  $\psi : D \rightarrow I$ .

The map  $x \mapsto \langle x, 0, 0 \rangle_A$  is a natural isomorphism between  $L$  and the principal ideal  $(p_1]$  of  $A$ ; this gives us a convex embedding of  $L$  into  $A$ . We identify  $L$  with its image, and regard  $L$  as a convex sublattice of  $A$  and therefore of  $K$ . So  $K$  is a convex extension of  $L$ . We have completed the construction of the bounded lattice  $K$  of Theorem 1.

### 3.2. Congruences of $K$

The proof in Part I heavily depended on the fact that we glued over ideals and dual ideals of which the building components were congruence-preserving extensions. This is no longer the case; however, a modification of Lemma 11 of Part I comes to the rescue.

A congruence  $\Omega$  of  $K$  can be described by four congruences,

- $\Omega_A$ , the restriction of  $\Omega$  to  $A$ ,
- $\Omega_{a,b}$ , the restriction of  $\Omega$  to  $L_{a,b}$ ,
- $\Omega_{c,d}$ , the restriction of  $\Omega$  to  $L_{c,d}$ ,
- $\Omega_B$ , the restriction of  $\Omega$  to  $B$ .

These congruences satisfy a number of conditions:

- (i)  $\langle 0, x, 0 \rangle_{L_{a,b}} \equiv \langle 0, y, 0 \rangle_{L_{a,b}} (\Omega_{a,b})$  iff  $\overline{\langle x, 1, 1 \rangle_B} \equiv \overline{\langle y, 1, 1 \rangle_B} (\Omega_B)$ , for  $x, y \in [a, b]$ .
- (ii)  $\langle 0, 0, x \rangle_A \equiv \langle 0, 0, y \rangle_A (\Omega_A)$  iff  $\langle x, c, x \wedge c \rangle_{L_{c,d}} \equiv \langle y, c, y \wedge c \rangle_{L_{c,d}} (\Omega_{c,d})$ , for  $x, y \in L$ .
- (iii)  $\langle x, 0, 0 \rangle_A \equiv \langle y, 0, 0 \rangle_A (\Omega_A)$  iff  $\langle x, a, x \wedge a \rangle_{L_{a,b}} \equiv \langle y, a, y \wedge a \rangle_{L_{a,b}} (\Omega_{a,b})$ , for  $x, y \in L$ .
- (iv)  $\langle 0, x\varphi, 0 \rangle_{L_{c,d}} \equiv \langle 0, y\varphi, 0 \rangle_{L_{c,d}} (\Omega_{c,d})$  iff  $\overline{\langle 1, x, x \rangle_B} \equiv \overline{\langle 1, y, y \rangle_B} (\Omega_B)$ , for  $x, y \in [a, b]$ .

Conversely, if we are given congruences  $\Omega_A$  on  $A$ ,  $\Omega_{a,b}$  on  $L_{a,b}$ ,  $\Omega_{c,d}$  on  $L_{c,d}$ ,  $\Omega_B$  on  $B$ , then by (i), we can define a congruence  $\Omega_U$  on  $U$ . By (ii), we can define a congruence  $\Omega_V$  on  $V$ . By (iii) and (iv), we can define a congruence  $\Omega_K$  on  $K$ .

Now it is clear that if we start with a congruence  $\Sigma$  of  $L$ , then we can define the congruences  $\Sigma_A$  on  $A$ ,  $\Sigma_{a,b}$  on  $L_{a,b}$ ,  $\Sigma_{c,d}$  on  $L_{c,d}$  componentwise, and  $\Sigma_B$  on  $B$  as in Section 2.5. Conditions (i)–(iii) trivially hold (since  $\Sigma_A$ ,  $\Sigma_{a,b}$ , and  $\Sigma_{c,d}$  are defined componentwise). Finally, (iv) holds if  $\Sigma$  is a  $\vec{\varphi}$ -congruence. So every  $\vec{\varphi}$ -congruence of  $L$  has an extension to  $K$ .

Let  $\Sigma$  be a congruence of  $L$  that extends to  $K$ . Since  $A$  is a congruence-preserving convex extension of  $L = [0_A, p_1]$ , further,  $L_{a,b}$  is a congruence-preserving extension of  $D_{a,b}$ , and  $L_{c,d}$  is a congruence-preserving extension of  $D_{c,d}$ , the congruence  $\Sigma$  uniquely extends to  $A$  as  $\Sigma_A$ , to  $L_{a,b}$  as  $\Sigma_{a,b}$ , and to  $L_{c,d}$  as  $\Sigma_{c,d}$ . Therefore,  $\Sigma$  uniquely extends to the intervals  $[\bar{r}, \bar{1}]$  and  $[\bar{p}, \bar{1}]$  of  $B$ , and so by Lemma 12 to  $B$ . We conclude that if a congruence  $\Sigma$  of  $L$  extends to  $K$ , then it extends uniquely.

To complete the proof, we prove that  $\varphi$  is algebraic. Define

$$\mathbf{p}(x) = (((((((x \wedge \langle a, a, b \rangle_{L_{a,b}}) \vee \langle b, a, a \rangle_{L_{a,b}}) \wedge p) \vee q_1) \wedge q_2) \\ \vee \langle d, c, c \rangle_{L_{c,d}}) \wedge \langle c, c, d \rangle_{L_{c,d}}) \vee p_2) \wedge p_1.$$

By Lemma 11,  $\mathbf{p}(x)$  behaves properly in  $B$ , while outside of  $B$ ,  $\mathbf{p}(x)$  is the same algebraic function as in Part I.

This completes the proof of Theorem 1.

#### 4. Proof of Theorem 2

We can assume, for  $i \in I$ , without loss of generality, that  $[a_i, b_i]$  and  $[c_i, d_i]$  are non-trivial intervals, that is,  $a_i < b_i$  and  $c_i < d_i$ . We can assume that  $|I| > 2$  by repeating an interval and a homomorphism, if necessary.

As in Section 3.1—*mutatis mutandis*—we take the building blocks:  $A = M_I \langle L \rangle$  (where  $M_I$  is the length two modular lattice with atoms  $p_i$ , for  $i \in I$ ),  $L_{a_i, b_i}$ ,  $B_i$ , and  $L_{c_i, d_i}$ , for  $i \in I$ . Let  $B$  be the dual discrete direct product of the  $B_i$ ,  $i \in I$ . Note that  $B$  does not have a zero in general. Now we proceed as in Section 7 of Part I: For  $i \in I$ , we glue  $L(a_i, b_i)$  to  $A$  (using the ideal  $[0, p_i]$  of  $A$ ) and at the same time glue it to  $B_i$ —and therefore, to  $B$ —using the dual ideal  $[\bar{r}_i]$  of  $B_i$  (therefore, of  $B$ ).

We can verify that  $L$  has the required properties as in Part I, except that now we rely on Lemma 12 when we utilize the properties of  $B$ . This completes the proof of Theorem 2.

Note that Theorem 2 generalizes both Theorems 2 and 3 of Part I. However, in spirit, it is closer to Theorem 3 of Part I. Indeed, in Theorem 2 of Part I, we consider a family  $[a_i, b_i]$ ,  $i < \alpha$ , of intervals of  $L$ , and the isomorphisms

$$\varphi_{i,j} : [a_i, b_i] \rightarrow [a_j, b_j], \quad \text{for } i, j < \alpha.$$

The isomorphisms are assumed to satisfy the natural “associativity” conditions, so they are independent.

In Theorem 3 of Part I, we are given a doubly indexed family  $[a_i^m, b_i^m]$ ,  $i < \alpha$ ,  $m < \mu$ , of intervals of  $L$ , and the isomorphisms

$$\varphi_{i,j}^m : [a_i^m, b_i^m] \rightarrow [a_j^m, b_j^m], \quad \text{for } i, j < \alpha, m < \mu.$$

The isomorphisms  $\varphi_{i,j}^m$ , for  $m$  fixed, are assumed to satisfy the natural “associativity” conditions, however, the isomorphisms  $\varphi_{i,j}^m$  and  $\varphi_{i,j}^n$  do not interact for  $m \neq n$ , with  $m, n < \mu$ . Similarly, in Theorem 2 of this paper, the homomorphisms  $\varphi_i$ , for  $i \in I$ , do not interact for  $i \neq j$ , with  $i, j \in I$ . In both cases, we attain this “independence” by forming the dual discrete direct product of the  $B_i$ , which results in a lattice without a zero element.

#### 5. Discussion

##### 5.1. Convex sublattices

Theorem 2 allows us to generalize Theorem 1 from intervals to convex sublattices. Let  $L$  be a bounded lattice, let  $U$  and  $V$  be convex sublattices of  $L$ , and let  $\varphi$  be a homomorphism from  $U$  onto  $V$ . We introduce a  $\vec{\varphi}$ -congruence, the partial algebra  $L_{\vec{\varphi}}$ , and a  $\vec{\varphi}$ -congruence-preserving extension as in Section 1.1, *mutatis mutandis*.

Let  $L$  be a lattice, let  $A, B \subseteq L$ , and let  $\varphi : A \rightarrow B$  be a map. We call the map  $\varphi$  *locally algebraic* iff for every  $[u, v] \subseteq A$ , there is a unary algebraic function  $\mathbf{p}(x)$  such that  $x\varphi = \mathbf{p}(x)$ , for all  $x \in [u, v]$ . For an interval  $A$ , locally algebraic is the same as algebraic.

Here is the generalization of Theorem 1:

**Theorem 1'.** *Let  $L$  be a bounded lattice, let  $U$  and  $V$  be convex sublattices of  $L$ , and let  $\varphi$  be a homomorphism from  $U$  onto  $V$ . Then  $L$  has a  $\vec{\varphi}$ -congruence-preserving convex extension into a lattice  $K$  such that  $\varphi$  is locally algebraic in  $K$ . In particular, the congruence lattice of the partial algebra  $L_{\vec{\varphi}}$  is isomorphic to the congruence lattice of the lattice  $K$ .*

Note that  $K$  is no longer claimed to be bounded because we obtain it from Theorem 2.

**Proof.** Let  $[a_i, b_i]$ ,  $i \in I$ , be the family of all subintervals of  $U$ . Let  $[c_i, d_i]$ ,  $i \in I$ , be the family of all corresponding subintervals of  $V$ , that is,  $c_i = a_i\varphi$  and  $d_i = b_i\varphi$ , for all  $i \in I$ , and define  $\varphi_i : [a_i, b_i] \rightarrow [c_i, d_i]$  as the restriction of  $\varphi$  to  $[a_i, b_i]$ , for  $i \in I$ . Now we get Theorem 1' by a straightforward application of Theorem 2.  $\square$

Of course, we can similarly generalize Theorem 2.

## 5.2. Fully invariant congruences

As usual, let us call a congruence  $\Theta$  *fully invariant* iff  $a \equiv b(\Theta)$  implies that  $a\alpha \equiv b\alpha(\Theta)$ , for any automorphism  $\alpha$ .

For a lattice  $L$ , let  $\text{Con}_{\text{inv}} L$  denote the lattice of fully invariant congruences of  $L$ , and let  $\text{Aut } L$  denote the set (group) of automorphisms of  $L$ .

For a bounded lattice  $L$ , we can apply Theorem 2 to  $I = \text{Aut } L$ ; for  $\alpha \in \text{Aut } L$ , let  $[a_\alpha, b_\alpha] = [c_\alpha, d_\alpha] = [0, 1]$ , and let  $\varphi_\alpha = \alpha$ .

**Theorem 3.** *Let  $L$  be a bounded lattice. Then  $L$  has a convex extension into a lattice  $K$  such that a congruence of  $L$  extends to  $K$  iff it is fully invariant and a fully invariant congruence of  $L$  extends uniquely to  $K$ . In particular,  $\text{Con}_{\text{inv}} L$  is isomorphic to  $\text{Con } K$ .*

## 5.3. The 1/3-Boolean triple construction

The reader may ask what is a 1/3-Boolean triple construction? For a lattice  $P$ , let us call the element  $\langle x, y, z \rangle \in P^3$  a *1/3-Boolean triple* iff

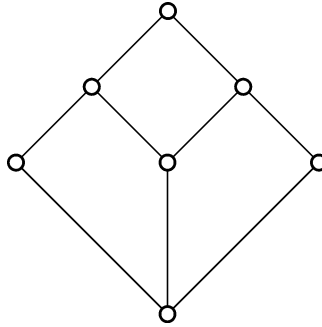
$$z = (z \vee x) \wedge (z \vee y).$$

Then instead of  $N_6$  of Fig. 1, we now get the lattice  $N_7$  of Fig. 4 (the dual of the seven-element semimodular but not modular lattice), and the 1/3-Boolean triples form a lattice isomorphic to  $N_7\langle P \rangle$ , using the notation of G. Grätzer and M. Greenberg [3], which, in turn, is isomorphic to  $N_7 \boxtimes P$ , using the notation of G. Grätzer and F. Wehrung [8].

## 5.4. Finite lattices

As a very simple application, we verify a result of R.P. Dilworth (first published in G. Grätzer and E.T. Schmidt [5]):



Fig. 4. The lattice  $N_7$ .

**Theorem 4.** *Every finite distributive lattice can be represented as the congruence lattice of a finite lattice.*

**Proof.** Let  $D$  be a finite distributive lattice, and let  $P = J(D)$  be the poset of join-irreducible elements of  $D$ . Let  $L$  be the Boolean lattice with atoms  $a_i$ ,  $i \in P$ . For  $i > j$  in  $P$ , define  $\varphi_{i,j}$ , the only homomorphism of  $[0, a_i]$  onto  $[0, a_j]$ . Apply Theorem 1 to these homomorphisms one at a time (or collectively apply Theorem 2) to construct a lattice  $K$  whose congruence lattice is isomorphic to  $D$ .  $\square$

Note that the results of Part I are not sufficient to provide this application: They only make intervals congruence equivalent. Here we want the intervals “congruence ordered.”

### 5.5. Congruence Lattice Problem

As discussed in Part I of this paper in Section 9, the fundamental unsolved problem in this field is the Congruence Lattice Problem: *Can every distributive algebraic lattice be represented as the congruence lattice of a lattice?*

The best method to attack this problem was found by E.T. Schmidt [9]. A recent result of F. Wehrung [11] shows the limitations of this method. See J. Tůma and F. Wehrung [10] for a review of related recent results.

It is natural to ask whether the method of E.T. Schmidt [9] can be combined with the results of this paper to solve the Congruence Lattice Problem. It is pointed out in J. Tůma and F. Wehrung [10] that this is not the case.

However, we can ask the following question.

**Problem 1.** Can every distributive algebraic lattice be represented as the congruence lattice of a partial algebra of the form  $F_0(\mathfrak{m})_\phi$ ?

( $F_0(\mathfrak{m})_\phi$  denotes the free lattice with zero on  $\mathfrak{m}$  generators.)

A positive answer to this question would partially answer Problem 2 of J. Tůma and F. Wehrung [10].

### 5.6. Bounds

Theorems 1 and 2 are proved in this paper only for bounded lattices. It is easy to see, however, that with minor technical changes we can prove them for arbitrary lattices  $L$  with zero. The lattice  $K$  we then obtain for Theorem 1 will have a zero but not necessarily a unit. The lattice  $K$  we then obtain for Theorem 2 may have neither zero nor unit.

In Theorem 1, we start with a bounded lattice  $L$ , and obtain a bounded lattice  $K$ . However, the construction does not preserve the bounds.

**Problem 2.** Can we strengthen Theorem 1 to obtain a  $\{0, 1\}$ -preserving (convex) extension?

In Theorem 2 the lattice constructed may not have a zero.

**Problem 3.** Can we strengthen Theorem 2 to obtain a bounded (convex) extension? Even stronger: a bounded (convex)  $\{0, 1\}$ -preserving extension?

### Acknowledgment

Again, just as in Part I, we would like to thank the referee for an unusually perceptive report, which resulted in a much improved paper.

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